Two-Phase Flow Through Porous Media

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In the context of a previously proposed new description for multiphase flow through porous media, we examine constitutive equations for $\mathbf{g}_i$, the force per unit volume which phase i exerts upon the pore walls and other adjacent fluids. The principle of material difference is used to determine the most general dependence of $\mathbf{g}_i$ upon saturation gradient for a nonoriented (no porosity gradient) permeable structure.

Three problems are solved and the results compared with analyses based upon our interpretation of Muskat's equations. The first concerns a semi-infinite porous column in air, the bottom of which is saturated with a liquid. The same saturation distribution is found with our approach as with Muskat's equations.

The analysis of the second problem indicates the relation between our relative permeability and that of Muskat.

The jump balances for mass and for momentum at a surface of discontinuity (or shock wave) with respect to volume averaged variables are derived. The significance of solutions in which such surfaces of discontinuity appear is discussed.

In the third problem, a plane surface of discontinuity with respect to local saturation moves through an infinite permeable structure. This is similar to the displacement of oil by an aqueous solution in a secondary recovery process. The results are comparable to those with Muskat's equations only in the limit where capillary pressure is neglected (the Buckley-Leverett case). When capillary pressure differs from zero, Muskat's equations do not admit a shock wave solution.

Within the context of single-phase flow of viscoelastic fluids in porous media, we have previously introduced the concept of local averaging of the equations of change, and we have examined the functional dependence of the force per unit volume which the fluid exerts on a porous structure (*1*, *2*). These ideas were illustrated by the solution of several one-dimensional problems of practical interest.

We have extended local averaging to each phase in multiphase flow through porous media; we have shown how the concept of capillary pressure is suggested by a local area average of the jump momentum balance at a phase interface, and we have discussed one approach to relative permeability for viscoelastic fluids (*3*). At the same time, we gave a plausible argument by which the inertial effects in the local averaged equation of motion could be neglected with respect to viscous effects for viscoelastic fluids, even though inertial effects cannot be neglected for the fluid in a particular pore during a Haines jump (*4*).

The results of our local averaging of the equations of continuity and of motion of phase i (*3*) can be readily summarized. Let us consider a particular point in a porous medium, and let there be associated with this point a closed surface S, the volume of which is V. This point may lie in the interior of S, but it is not essential that it do so. This same surface S may be associated with every point in the porous medium by a translation without rotation. For example, if S is a unit sphere, the center of which coincides with the point considered, we may center on each point in the porous medium (no matter whether it is in one of the fluid phases or within the solid phase) a unit sphere. Let V_i denote the region of the pores which contains phase i in the interior of S; the volume and shape of V_i in general will change from point to point in the porous medium. The closed boundary surface of V_i, S_i, is the sum of S_{ei}, S_{fi}, and S_{mi}. S_{ei} is that portion of S_i which coincides with S, S_{fi} with the pore walls and S_{mi} with the boundaries of other moving and deforming phases present. The local averages of the equations of continuity and of motion for phase i are, respectively

$$\frac{\partial \overline{\rho}^{\,i}}{\partial t} + \operatorname{div}\overline{(\rho \mathbf{v})}^{\,i} = 0 \tag{1}$$

and

$$\frac{\partial \overline{\rho \mathbf{v}}^{\,i}}{\partial t} + \operatorname{div}\overline{(\rho\, \mathbf{v}\, \mathbf{v})}^{\,i} = \operatorname{div}\overline{\mathbf{T}}^{\,i} + \overline{\rho\, \mathbf{f}}^{\,i} + \frac{1}{V}\int_{S_{mi}+S_{fi}} \mathbf{T}\cdot\mathbf{n}\, dS \tag{2}$$

If B is any scalar, vector, or tensor, we define

$$\overline{B}^{i} \equiv \frac{1}{V}\int_{V_i} B\, dV \tag{3}$$

In words, $\overline{B}^{i}$ is an average over V of a quantity B associated with phase i.

In order to simplify our discussion, we limit ourselves here to the movement of two incompressible fluids through a porous structure, though much of what we say may be readily generalized to the simultaneous flow of three or more phases. Equation (1) consequently reduces to

$$\psi \frac{\partial s_i}{\partial t} + \operatorname{div} \overline{\mathbf{v}}^{i} = 0 \tag{4}$$

Here ψ is the porosity

$$\psi \equiv \frac{1}{V}(V_1 + V_2) \tag{5}$$

and s_i is the local saturation of phase i:

$$s_i \equiv \frac{V_i}{V_1 + V_2} \tag{6}$$

When we neglect all inertial effects in the local volume averaged equation of motion (3) and represent the external force per unit mass $\mathbf{f}$ by a potential φ

$$\mathbf{f} = -\nabla\varphi \tag{7}$$

we may write Equation (2) as

$$\nabla(\overline{\mathcal{P} - p_0}^{i}) - \operatorname{div}(\overline{\mathbf{T} + p\,\mathbf{I}}^{i}) - \frac{1}{V}\int_{S_{mi}+S_{fi}} [\mathbf{T} + (p_0 - \rho_i\,\varphi)\mathbf{I}]\cdot\mathbf{n}\; ds = 0 \tag{8}$$

Here we define

$$\overline{\mathcal{P}}^{i} \equiv \overline{p}^{i} + \rho_i\,\overline{\varphi}^{i} \tag{9}$$

A constant reference or ambient pressure p_0 is introduced here in order that we may identify $\mathbf{g}_i$

$$\mathbf{g}_i \equiv -\frac{1}{V}\int_{S_{mi}+S_{fi}} [\mathbf{T} + (p_0 - \rho_i\,\varphi)\mathbf{I}]\cdot\mathbf{n}\; dS \tag{10}$$

as the force per unit volume which phase i exerts upon the pore walls and the other fluid phases contained within S beyond the hydrostatic force and beyond any force attributable to the ambient pressure. This force per unit volume $\mathbf{g}_i$ is entirely assignable to the motion of phase i.

By a local area average of the jump momentum balance at the fluid-fluid phase interface, we suggested that the capillary pressure p_c (3)

$$p_c \equiv \langle p\rangle^1 - \langle p\rangle^2 = \langle p - p_0\rangle^1 - \langle p - p_0\rangle^2 \tag{11}$$

may be represented by

$$p_c = \frac{\sigma}{L}\, p_c^{+}\,(s_1, \theta, \psi) \tag{12}$$

Here p_c^{+} is a dimensionless function of the local saturation of one of the phases, of the contact angle θ between the fluid-fluid phase interface and the solid composing the pore walls, and of the porosity ψ. We define

$$\langle p\rangle^i \equiv \frac{1}{V_i}\int_{V_i} p\; dV = \frac{\overline{p}^{i}}{\psi\, s_i} \tag{13}$$

This average over phase i of pressure is believed to correspond closely to the averaged pressure an experimentalist might measure for phase i.

In (3) we introduced a resistance transformation $\mathbf{K}_i$ by requiring that

$$-\operatorname{div}(\overline{\mathbf{T} + p\;\mathbf{I}}^{i}) + \mathbf{g}_i = \mathbf{K}_i\cdot\overline{\mathbf{v}}^{i} \tag{14}$$

It is more natural to discuss the two terms of the left of this equation separately as we do in the next two sections.

Thereafter, we discuss two one-dimensional problems in which all averaged variables are continuous functions of position. After deriving the balance equations at a surface singular with respect to one or more of the averaged variables, we conclude with an analysis of a displacement problem in an infinite medium.

FORCE EXERTED BY PHASE *i* UPON THE SURROUNDING PHASES

We follow closely here our discussion of the force exerted upon the solid in single phase flow (2). Consequently, some of the details given there are not repeated.

For the moment, let us assume that $\mathbf{g}_i$ is a function of the difference between the local volume-averaged velocity of phase i, $\overline{\mathbf{v}}^{i}$, and the local volume-average velocity of the solid $\overline{\mathbf{u}}$:

$$\mathbf{g}_i = \hat{\mathbf{g}}_i\,(\overline{\mathbf{v}}^{i} - \overline{\mathbf{u}}) \tag{15}$$

Functional dependence upon several scalars such as porosity, a characteristic length of the porous medium, the characteristic viscosities, and the characteristic times of the various phases is understood in this expression. As a result of this explicit dependence upon both the characteristic viscosities and characteristic times of the fluids present, this development is designed to apply both to Newtonian fluids and to viscoelastic fluids such as the incompressible Noll simple fluid (5, 6). We define

$$\overline{\mathbf{u}} \equiv \frac{1}{V}\int_{V-V_1-V_2} \mathbf{v}\; dV \tag{16}$$

By the principal of material frame indifference (7, p. 34; 8, p. 44; 2, see force exerted . . .) the function $\hat{\mathbf{g}}_i$ must be such that for any time dependent orthogonal transformation $\mathbf{Q}(t)$

$$\mathbf{Q}(t)\cdot\hat{\mathbf{g}}_i(\overline{\mathbf{v}}^{i} - \overline{\mathbf{u}}) = \hat{\mathbf{g}}_i\left(\mathbf{Q}(t)\cdot\{\overline{\mathbf{v}}^{i} - \overline{\mathbf{u}}\}\right) \tag{17}$$

This means that $\hat{\mathbf{g}}_i$ is an isotropic function (8, p. 22). By a representation theorem for a vector valued isotropic function of one vector (8, p. 35), we may write

$$\mathbf{g}_i = \hat{\mathbf{g}}_i(\overline{\mathbf{v}}^{i} - \overline{\mathbf{u}}) = R_i\,\{\overline{\mathbf{v}}^{i} - \overline{\mathbf{u}}\} \tag{18}$$

It is to be understood here that the resistance coefficient R_i is a function of the magnitude of the local volume averaged velocity of phase i relative to the local volume averaged velocity of the solid $|\overline{\mathbf{v}}^{i} - \overline{\mathbf{u}}|$ as well as the various scalars mentioned previously. A discussion of functional dependence given previously (3, see the functional dependence . . .) may be applied here to R_i.

One would not expect Equation (18) to be valid for a porous structure in which porosity ψ is a function of position. For such a structure Equation (15) must be altered to include a dependence upon additional vector and possible tensor quantities. For example, one might postulate a dependence of $\mathbf{g}_i$ upon the local porosity gradient as well as upon $\overline{\mathbf{v}}^{i} - \overline{\mathbf{u}}$. By the same reasoning, if a saturation gradient exists in a permeable structure of uniform porosity, one should, in general, expect a dependence of $\mathbf{g}_i$ upon the local saturation gradient as well as upon $\overline{\mathbf{v}}^{i} - \overline{\mathbf{u}}$:

$$\mathbf{g}_i = \hat{\mathbf{g}}_i(\overline{\mathbf{v}}^{i} - \overline{\mathbf{u}},\ \nabla s_1) \tag{19}$$

The function $\hat{\mathbf{g}}_i$ again must be isotropic by the principle of frame indifference:

$$\mathbf{Q}(t) \cdot \hat{\mathbf{g}}_i(\overline{\mathbf{v}}^{i} - \overline{\mathbf{u}}, \nabla s_1) = \hat{\mathbf{g}}_i\left(\mathbf{Q}(t) \cdot \{\overline{\mathbf{v}}^{i} - \overline{\mathbf{u}}\}, \mathbf{Q}(t) \cdot \nabla s_1\right) \tag{20}$$

By representation theorems of Spencer and Rivlin (*9*, section 7) and of Smith (*10*), the most general polynomial *isotropic vector function* of two vectors is of the form

$$\mathbf{g}_i = \varphi_{(1)i}\{\overline{\mathbf{v}}^{i} - \overline{\mathbf{u}}\} + \varphi_{(2)i}\ \nabla s_1 \tag{21}$$

where $\varphi_{(1)i}$ and $\varphi_{(2)i}$ are scalar valued polynomials in $|\overline{\mathbf{v}}^{i} - \overline{\mathbf{u}}|$, $|\nabla s_1|$, and $(\overline{\mathbf{v}}^{i} - \overline{\mathbf{u}}) \cdot \nabla s_1$. We expect $\varphi_{(2)i} = 0$ for $|\overline{\mathbf{v}}^{i} - \overline{\mathbf{u}}| = 0$ in order that $\mathbf{g}_i = 0$ in this limit.

An even more complicated expression than Equation (21) should be expected for nonuniform porous structures.

DIVERGENCE OF THE EXTRA STRESS

For the moment, let us limit ourselves to incompressible Newtonian fluids, and let us assume that viscosity is a constant, at least locally in V_i. By a development very similar to that given in a previous discussion of single-phase flow (see *2*, divergence of . . .), we have that

$$\overline{(\mathbf{T} + p\ \mathbf{I})}^{i} = \mu_i\,[\nabla\overline{\mathbf{v}}^{i} + (\nabla\overline{\mathbf{v}}^{i})^{t}\,] \tag{22}$$

If we assume that Equation (18) represents the force per unit volume which fluid *i* exerts upon the surrounding phases, Equation (8) becomes for such a fluid

$$\nabla\overline{(\mathcal{P} - p_0)}^{i} - \mu_i\,[\mathrm{div}(\nabla\overline{\mathbf{v}}^{i}) + \nabla(\mathrm{div}\overline{\mathbf{v}}^{i})] + R_i\ \overline{\mathbf{v}}^{i} = 0 \tag{23}$$

Comparing this with Equation (27) of (*3*), we see *that the second term on the left of Equation* (23) *is new.* The boundary conditions which we require the local volume averaged velocity field $\overline{\mathbf{v}}^{i}$ to satisfy have a bearing on the importance of this term.

Consider flow through a porous structure which is bounded by impermeable walls, such as a bed of sand packed in a pipe or a layer of sandstone bounded by relatively impermeable rock. As we advance into the impermeable boundary from the porous structure, the local volume averaged velocity of phase *i*, $\overline{\mathbf{v}}^{i}$, decreases, since less of the averaging surface *S* intersects the porous structure through which *i* is flowing. At some distance ϵ (which depends upon the averaging surface chosen) inside the impermeable wall, $\overline{\mathbf{v}}^{i}$ becomes zero. This is a boundary condition which must be satisfied by the local volume averaged velocity field for phase *i*.

Consider two-phase flow through a bed of sand packed *in a pipe such that the local saturation for each phase is independent of position.* We visualize the local volume averaged velocity field for each phase to be one dimensional. Yet a solution in which only the axial component of *velocity differs from zero cannot satisfy both the bound*ary condition in the impermeable wall and Equation (23) when the second term on the left is neglected [see *3*, Equation (27)]. A primary advantage of Equation (23) when all terms are retained is that a nondimensional solution for this geometry can satisfy such a boundary condition.

We could solve several problems for two-phase flow of incompressible Newtonian fluids in a porous structure under conditions such that the local saturations were independent of position. These problems would be very similar to those solved previously for single-phase flow (*2*). We could also construct order-of-magnitude arguments analogous to those given for single-phase flow (*2*, see importance of . . . and viscoelastic fluids). The effect of these problems and of these order-of-magnitude arguments would be to suggest that, so long as one is not concerned with the volume averaged velocity distribution in the immediate vicinity of a solid boundary, it does not appear important to satisfy boundary conditions on the tangential components of volume averaged velocity or to include the effects of the second term on the left of Equations (8) and (23). For this reason, we shall hereafter approximate Equation (8) as

$$\nabla\overline{(\mathcal{P} - p_0)}^{i} + \mathbf{g}_i = 0 \tag{24}$$

When $\mathbf{g}_i$ *is represented by* Equation (18), we have essentially the same result suggested previously [*3*, Equation (27)].

In what follows, we consider two example problems, where our primary interest will be to compare solutions using Equation (24) and either Equation (18) or (21) with solutions based upon Muskat's extension of Darcy's law to multiphase flow in porous media (*11*, *12*, p. 216):

$$\nabla\langle\mathcal{P} - p_0\rangle^{i} + L_i\ \overline{\mathbf{v}}^{i} = 0 \tag{25}$$

[In order to make the comparison as obvious as possible, we have interpreted Muskats equation in terms of our notation. Comparing Equation (4) with his corresponding equation (*12*, Equation 9.2.2.5), we identify Muskat's "seepage velocity" with $\overline{\mathbf{v}}^{i}$. The quantity $\langle\mathbf{v}\rangle^{i} = \dfrac{\overline{\mathbf{v}}^{i}}{\psi\ s_i}$ may be thought of as the Dupuit-Forchheimer pore velocity (*12*, p. 115). Comparison of Equation (11) with Muskat's *capillary pressure relationship* (*12*, Equation 9.2.2.7) suggests the use of $\nabla\langle\mathcal{P} - p_0\rangle^{i}$ in Equation (25).]

A STATIC EXPERIMENT

Starting with an initially fluid free porous medium in air, we allow the experiment illustrated in Figure 1 to come to equilibrium. We assume that the porous medium extends to infinity in the positive *z* direction, and we neglect any mass transfer of the fluid into the air. Our experiment is very simply described by requiring as boundary conditions

$$\text{at } z = 0:\quad s_1 = 1,\quad \langle\mathcal{P} - p_0\rangle^{1} = 0 \tag{26}$$

Since the origin for the *z* axis and the ambient pressure p_0 are both arbitrary, there is no difficulty in assigning these boundary conditions. We seek the local saturation of each phase s_i and $\langle\mathcal{P} - p_0\rangle^{i}$ as functions of *z* subject to boundary conditions (*26*).

Equation (24) together with either Equation (18) or

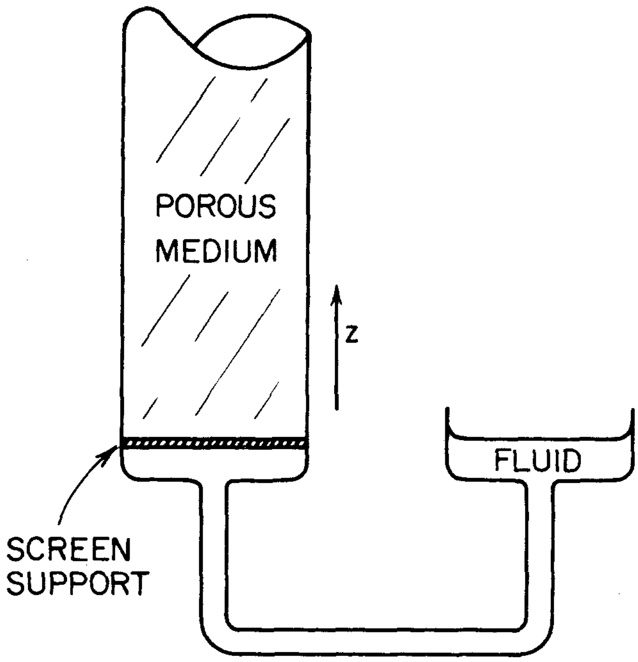

Fig. 1. A static experiment.

Equation (21) [we require $\varphi_{(2)i} = 0$ for $|\overline{v} - \overline{u}| = 0$] reduces to

$$\nabla (\psi\ s_i <\mathcal{P} - p_0>^i) = 0 \tag{27}$$

which implies that

$$\psi\ s_1 <\mathcal{P} - p_0>^1 = C_{(1)}, \quad \psi\ s_2 <\mathcal{P} - p_0>^2 = C_{(2)} \tag{28}$$

In view of boundary conditions (26) and the fact that $s_2 = 1 - s_1$, Equations (28) imply

$$<\mathcal{P} - p_0>^1 = <\mathcal{P} - p_0>^2 = 0 \tag{29}$$

Equations (9), (11), (12), and (29) give us

$$\rho_2 <\varphi>^2 - \rho_1 <\varphi>^1 = p_c = \frac{\sigma}{L}\ p_c^+ (\theta, s_1, \psi) \tag{30}$$

which is to be satisfied by s_1 as a function of z.

If we take the approach of Muskat (*11, 12,* p. 216), from Equation (25) we find

$$<\mathcal{P} - p_0>^1 = C_{(3)}, \quad <\mathcal{P} - p_0>^2 = C_{(4)} \tag{31}$$

where $C_{(3)}$ and $C_{(4)}$ are constants to be determined from the boundary conditions. Equation (26_2) implies immediately that $C_{(3)} = 0$. Equation (26_1) requires

$$\text{at} \quad z = 0: \quad <\mathcal{P} - p_0>^2 = 0 \tag{32}$$

since there is no phase 2 present over which to average pressure. We consequently have that $C_{(4)} = 0$ as well, so that Equations (28) and (29) follow as before. For this problem, there is no difference between the approach of Muskat and that recommended here.

A RELATIVE PERMEABILITY EXPERIMENT

We now wish to consider steady state, incompressible, two-phase flow through a permeable structure of uniform porosity ψ bounded by an infinitely long impermeable cylindrical surface of radius r_0. The development applies equally well to Newtonian and viscoelastic fluids. In order to simplify the computations, we neglect gravitational effects. We take as our boundary conditions

$$\text{at} \quad r = r_0: \quad \overset{i}{\overline{v}}_r = 0 \tag{33}$$

$$\text{at} \quad r = 0, \quad z = 0: \quad \overset{2}{\overline{v}}_z = \overset{2}{\overline{v}}_{z(0)} \tag{34}$$

and

$$\text{at} \quad r = r_0,\ \theta = 0,\ z = 0: \quad <p - p_0>^1 = <p - p_0>^1_{(0)} \tag{35}$$

We seek a solution of such a form that the saturation of phase 1 is independent of position

$$s_1 = s_{(0)} = \text{a constant} \tag{36}$$

where $s_{(0)}$ is to be determined.

Let us see if there is a solution such that

$$\overset{i}{\overline{v}}_r = \overset{i}{\overline{v}}_\theta = 0, \quad \overset{i}{\overline{v}}_z = \overset{i}{\overline{v}}_{z(0)} \tag{37}$$

In this way we satisfy automatically the equation of continuity for each phase, Equation (4), as well as boundary conditions (33) and (34).

In order to simplify the computations, we introduce the following dimensionless variables:

$$P^i \equiv \frac{<p - p_0>^i \sqrt{k}}{\mu_2\ \overset{2}{\overline{v}}_{z(0)}}$$

$$u^i \equiv \frac{\overset{i}{\overline{v}}_z}{\overset{2}{\overline{v}}_{z(0)}}, \quad \eta \equiv \frac{z}{\sqrt{k}}, \quad p_c^+ = \frac{\sqrt{k}}{\sigma}\ p_c \tag{38}$$

The permeability of the pore structure to a single Newtonian phase is denoted by k. Equations (11), (12), (24), and either (18) or (21) consequently yield as the system of equations to be solved

$$\frac{dP^1}{d\eta} = - \frac{N_{(1)}\ u^1}{\psi\ s_{(0)}\ \beta_{1(0)}} \tag{39}$$

$$\frac{dP^2}{d\eta} = - \frac{1}{\psi\ [1 - s_{(0)}]\ \beta_{2(0)}} \tag{40}$$

and

$$P^1 - P^2 = N_{(2)}\ p^+_{c(0)} \tag{41}$$

where

$$N_{(1)} \equiv \mu_1/\mu_2, \quad N_{(2)} \equiv \frac{\sigma}{\mu_2\ \overset{2}{\overline{v}}_{z(0)}}, \quad \beta_{i(0)} = \frac{\mu_i}{k\ R_{i(0)}} \tag{42}$$

Here β_i may be referred to as the relative permeability for phase i; by $\beta_{i(0)}$ and $R_{i(0)}$ we mean β_i and R_i evaluated for $s_1 = s_{(0)}$, $\overset{1}{\overline{v}}_z = \overset{1}{\overline{v}}_{z(0)}$, and $\overset{2}{\overline{v}}_z = \overset{2}{\overline{v}}_{z(0)}$ (see 3, the functional dependence . . .). [If Equation (21) is used rather than Equation (18), $R_{i(0)}$ should be replaced by $\varphi_{(1)i}$ evaluated at $s_1 = s_{(0)}$, $\overset{1}{\overline{v}}_z = \overset{1}{\overline{v}}_{z(0)}$, and $\overset{2}{\overline{v}}_z = \overset{2}{\overline{v}}_{z(0)}$.] In the same way, $p^+_{c(0)}$ denotes p^+_c evaluated for $s_1 = s_{(0)}$.

Boundary condition (35) now takes the form

$$\text{at} \quad \eta = 0: \quad P^1 = P^1_{(0)} \equiv \frac{<p - p_0>^1_{(0)} \sqrt{k}}{\mu_2\ \overset{2}{\overline{v}}_{z(0)}} \tag{43}$$

Since $p^+_{c(0)}$ is a constant, from Equation (41) we have

$$\frac{dP^1}{d\eta} = \frac{dP^2}{d\eta} \tag{44}$$

Consequently, from Equations (39), (40), and (44)

$$u^1 = \frac{s_{(0)}\ \beta_{1(0)}}{[1 - s_{(0)}]\ \beta_{2(0)}\ N_{(1)}} \tag{45}$$

By means of Equations (40) and (41), we arrive at

$$P^2 = -\frac{\eta}{\psi[1 - s_{(0)}]\ \beta_{2(0)}} + C_{(1)} \tag{46}$$

and

$$P^1 = -\frac{\eta}{\psi\,[1 - s_{(0)}]\ \beta_{2(0)}} + C_{(1)} + N_{(2)}\ p^+{}_{c(0)} \tag{47}$$

The quantity p_0 is an arbitrary constant which we may specify by taking

$$\text{at}\quad \eta = 0:\quad P^2 = 0 \tag{48}$$

From Equation (46), we see that this implies

$$C_{(1)} = 0 \tag{49}$$

Finally, boundary condition (43), together with Equations (47) and (49), yields

$$P^1{}_{(0)} = N_{(2)}\ p^+{}_{c(0)} \tag{50}$$

which is to be used together with capillary pressure function to determine $s_{(0)}$.

If the relative permeabilities β_1 and β_2 are to be determined from this experiment, P^1 and P^2 must be measured as a function of η. We may then evaluate $\beta_{1(0)}$ and $\beta_{2(0)}$ by means of Equations (39) and (40). The experiment must be repeated for a variety of $\overset{2}{\overline{v}}_{z(0)}$ and $<p - p_0>^1$ in Equations (34) and (35).

If we choose to use the approach of Muskat for Newtonian fluids, from Equation (25) we have in the same notation

$$\frac{dP^1}{d\eta} = -\frac{N_{(1)}\ u^1}{\alpha_{1(0)}} \tag{51}$$

and

$$\frac{dP^2}{d\eta} = -\frac{1}{\alpha_{2(0)}} \tag{52}$$

Here we use α_i for Muskat's relative permeability of phase i. The argument then goes through as before to conclude that

$$u^1 = \frac{\alpha_{1(0)}}{\alpha_{2(0)}\ N_{(1)}} \tag{53}$$

$$P^2 = -\frac{\eta}{\alpha_{2(0)}} \tag{54}$$

and

$$P^1 = -\frac{\eta}{\alpha_{2(0)}} + N_{(2)}\ p^+{}_{c(0)} \tag{55}$$

Equation (50) applies for this case as well to determine $s_{(0)}$.

The following relationship holds between the Muskat relative permeability α_i and that recommended here, β_i:

$$\alpha_i = \psi\ s_i\ \beta_i \tag{56}$$

BALANCE EQUATIONS AT A SINGULAR SURFACE

Buckley and Leverett (*13, 12*, p. 224; see also appendix) found a solution to Equation (25), Muskat's equation, in which a surface of discontinuity or shock wave moves through the porous medium.

A surface of discontinuity with respect to one or more local volume averaged variables seems to have little physical significance. The volume averaging in Equation (3) insures that a surface of discontinuity with respect to $\overset{i}{\overline{B}}$ cannot appear. Yet, because Equations (18), (21), and (24) represent approximations to reality, solutions in which singular surfaces appear may exist. To the extent that these equations are useful approximations to reality, we would expect that such solutions might be worthwhile representations of real phenomena.

Our purpose here is to indicate the restrictions which should be placed upon the volume averaged variables at a singular surface. We seek parallels to the jump mass and jump momentum balances at singular surfaces which are required by the postulate that mass is conserved for a body and by the postulate of Euler's first law (Newton's second law), (*14*, p. 526). The discussion here is similar to that given previously for phase interfaces (*15*).

In continuum mechanics, both the jump mass balance and the differential equation continuity are derived from one basic postulate: that mass is conserved for every body whether or not the body is split by singular surfaces. Here we have a differential equation which expresses conservation of mass at each point in a porous medium in terms of the local volume averaged variables. In order to derive a jump mass balance at a surface which is singular with respect to one or more of the local volume averaged variables, it is necessary that we make a postulate of mass conservation for systems composed of artificial particles which move with some averaged velocity of the fluid. This is essentially our approach below, both for the jump mass and jump momentum balances.

In order to keep things as simple as possible, let us restrict ourselves to incompressible materials and write Equation (4) as

$$\frac{\partial \psi s_i}{\partial t} + \mathrm{div}\overset{i}{\overline{\mathbf{v}}} = 0 \tag{57}$$

Let us consider an integral of this equation over a moving and deforming region $R_{(c)i}$ of phase i in a permeable structure such that the boundary of the region moves with a velocity $<\mathbf{v}>^i \equiv \dfrac{\overset{i}{\overline{\mathbf{v}}}}{\psi s_i}$ and such that ψ, s_i, $\overset{i}{\overline{\mathbf{v}}}$, and their derivatives are continuous in this region:

$$\int_{R_{(c)i}} \left\{ \frac{\partial\, \psi s_i}{\partial\, t} + \mathrm{div}\overset{i}{\overline{\mathbf{v}}} \right\} dR = 0 \tag{58}$$

The transport theorem generalized for nonmaterial regions (*14*, p. 347) may be used to rewrite Equation (58) as

$$\frac{d}{d\ t} \int_{R_{(c)i}} \psi s_i\ dR = 0 \tag{59}$$

Equation (59) indicates that $R_{(c)i}$ is a region containing a constant volume of phase i.

As a result, it seems entirely reasonable to say that for any region R_i, the boundaries of which move with a velocity $<\mathbf{v}>^i$, must be a region containing a constant volume of phase i and consequently must obey

$$\frac{d}{d\ t} \int_{R_i} \psi s_i\ dR = 0 \tag{60}$$

By a simple extension of Kotchine's theorem (*14*, Equation 193.3), we have

$$\left[\overset{i}{\overline{\mathbf{v}}} \cdot \boldsymbol{\xi} - \psi s_i\, V_{(\xi)}\right] = 0 \tag{61}$$

By the bold face brackets in this equation we mean the jump across the singular surface of the quantity enclosed by the brackets; $\boldsymbol{\xi}$ is the unit normal vector to the singular surface and $V_{(\xi)}$ is the speed of displacement of the singular surface (*14*, p. 499).

In the same way, we may integrate Equation (24) over the moving and deforming region $R_{(c)i}$; after an application of the generalized Green's transformation (*14*, p. 815), we have

$$\int_{S_{(c)i}} \overset{i}{\overline{(\mathcal{P} - p_0)}}\ \mathbf{n}\ dS + \int_{R_{(c)i}} \mathbf{g}_i\ dR = 0 \tag{62}$$

Here $S_{(c)i}$ denotes the closed bounding surface of $R_{(c)i}$. Again, it seems perfectly reasonable to postulate that this statement of a force balance is applicable to any region R_i containing a constant volume of phase i even though the region may be split by a number of surfaces which are singular with respect to one or more of the averaged variables:

$$\int_{S_i} \overset{i}{\overline{(\mathcal{P} - p_0)}}\ \mathbf{n}\ dS + \int_{R_i} \mathbf{g}_i\ dR = 0 \tag{63}$$

Again, a simple extension of Kotchine's theorem allows us to conclude that at a singular surface

$$\left[\overset{i}{\overline{(\mathcal{P} - p_0)}}\right] = [\psi s_i < \mathcal{P} - p_0 >^i] = 0 \tag{64}$$

This represents a jump momentum balance at a surface which is singular with respect to one or more local volume averaged variables.

It can be readily shown the jump mass balance, Equation (61), is consistent with special cases given previously (for example, *12*, p. 226).

As we pointed out at the beginning of this section, a discontinuity in an averaged variable has doubtful significance. These doubts are reinforced here by the fact that Equations (60) and (63) or Equations (61) and (64) represent additional postulates which one must make beyond the fundamental postulates of mechanics.

A DISPLACEMENT PROBLEM

Let us again consider two-phase flow of incompressible fluids through a permeable structure of uniform porosity which is bounded by an infinite impermeable cylinder of radius r_0. The development applies equally well to Newtonian fluids and to viscoelastic fluids; the only difference is the functional dependence of R_i (*3*, see the functional dependence . . .). Initially, the saturation of phase 1 is uniform throughout the porous medium. We then attempt to flush all or at least a portion of phase 1 (oil) out of the porous medium with phase 2 (an aqueous solution).

To be more specific, let us describe our problem by following a set of boundary conditions:

$$\text{for all } t, \text{ at } r = r_0\text{: } \overset{i}{\overline{v_r}} = 0 \tag{65}$$

$$\text{for } t > 0, \text{ at } r = 0,\ z = 0\text{: } \overset{2}{\overline{v_z}} = \overset{2}{\overline{v}}_{z(0)},\ \overset{1}{\overline{v_z}} = 0 \tag{66}$$

$$\text{at } t = 0, \text{ for all } z > 0\text{: } s_1 = s_{(\infty)} \tag{67}$$

and

$$\text{for } t > 0, \text{ as } z \to \infty\text{: } s_1 \to s_{(\infty)} \tag{68}$$

Here $\overset{2}{\overline{v}}_{z(0)}$ and $s_{(\infty)}$ are known constants.

Let us attempt to find a solution such that the averaged velocity distribution in cylindrical coordinates has the form

$$\overset{i}{\overline{v_r}} = \overset{i}{\overline{v_\theta}} = 0,\ \overset{i}{\overline{v_z}} = \overset{i}{\overline{v_z}}(z, t) \tag{69}$$

The form of the problem is somewhat simpler, if we introduce the dimensionless variables defined by Equations (38). In addition, we define a dimensionless time Θ by

$$\Theta \equiv \frac{t\ \overset{2}{\overline{v}}_{z(0)}}{\sqrt{k}} \tag{70}$$

In terms of these variables and for a velocity distribution of the form of Equation (69), Equations (4), (11), (12), (18), and (24) yield

$$\psi \frac{\partial s_1}{\partial \Theta} + \frac{\partial u^1}{\partial \eta} = 0 \tag{71}$$

$$-\psi \frac{\partial s_1}{\partial \Theta} + \frac{\partial u^2}{\partial \eta} = 0 \tag{72}$$

$$\frac{\partial(\psi s_1 P^1)}{\partial \eta} + \frac{N_{(1)}\, u^1}{\beta_1} = 0 \tag{73}$$

$$\frac{\partial(\psi\{1 - s_1\}\, P^2)}{\partial \eta} + \frac{u^2}{\beta_2} = 0 \tag{74}$$

and

$$P^1 - P^2 = N_{(2)}\ p_c^{+} \tag{75}$$

Let us specialize still further and look for a solution in which a surface singular with respect to the saturation of phase 1 moves with a constant speed of displacement $V_{(\xi)}$ in the positive z direction. This suggests introducing the change of variable

$$\chi \equiv \eta - c\,\Theta \tag{76}$$

where

$$c \equiv V_{(\xi)} / \overset{2}{\overline{v}}_{z(0)} \tag{77}$$

We may now look for a solution such that

$$\text{for } \chi < 0\text{: } s_1 = s_{(0)},\ u^1 = 0,\ u^2 = 1 \tag{78}$$

and

$$\text{for } \chi > 0\text{: } s_1 = s_{(\infty)},\ u^1 = u^1_{(\infty)},\ u^2 = u^2_{(\infty)} \tag{79}$$

Here $s_{(0)}$, $u^1_{(\infty)}$, and $u^2_{(\infty)}$ are constants to be determined. A solution which has the form of Equations (78) and (79) automatically satisfies boundary conditions (65 to 68) as well as the two equations of continuity, Equations (71) and (72).

Let us consider the region $\chi < 0$ first. In view of Equation (78_2), Equation (73) reduces to

$$\beta_{1(0)} = 0 \tag{80}$$

By a subscript (0) we will mean hereafter a quantity associated with the region $\chi < 0$. The relative permeability $\beta_{1(0)}$ is a constant, since saturation and velocity are constants for both phases. Equation (80) may be used together with data for the relative permeability of phase 1 as a function of saturation to determine $s_{(0)}$. Since, experimentally, relative permeability is generally found to go to zero for a finite value of saturation (*16*), $s_{(0)}$ will usually be nonzero [for the relation between the relative permeability of phase i discussed here, β_i, and the usual relative permeability for phase i, α_i, see Equation (56)].

In view of Equations (78_1) and (78_3), Equation (74) may be integrated to yield

$$P^2_{(0)} = -\frac{\chi}{\psi\,[1 - s_{(0)}]\ \beta_{2(0)}} + C_{(1)} \tag{81}$$

Here $C_{(1)}$ is a constant to be determined. Referring to our discussion of $\beta_{1(0)}$, we see that $\beta_{2(0)}$ must be a constant as well; its value is known once we have been given relative permeability data for phase 2 and once $s_{(0)}$ has been determined by using Equation (80). Equation (75) allows us to determine $P^1_{(0)}$ from Equation (81) as

$$P^1_{(0)} = -\frac{\chi}{\psi\,[1 - s_{(0)}]\,\beta_{2(0)}} + C_{(1)} + N_{(2)}\,p^+{}_{c(0)} \quad (82)$$

By Equation (12), the dimensionless capillary pressure $p^+{}_{c(0)}$ must be a constant as the result of our assumption that $s_{(0)}$ is a constant in Equation (78_1).

Going now to the region $\chi > 0$, we find that Equations (73), (74), and (75) become

$$\frac{d\,P^1_{(\infty)}}{d\chi} + \frac{N_{(1)}\,u^1_{(\infty)}}{\psi s_{(\infty)}\,\beta_{1(\infty)}} = 0 \quad (83)$$

$$\frac{d\,P^2_{(\infty)}}{d\chi} + \frac{u^2_{(\infty)}}{\psi\,[1 - s_{(\infty)}]\,\beta_{2(\infty)}} = 0 \quad (84)$$

and

$$P^1_{(\infty)} - P^2_{(\infty)} = N_{(2)}\,p^+{}_{c(\infty)} \quad (85)$$

By reasoning similar to that given for the region $\chi < 0$, we have that $\beta_{1(\infty)}$, $\beta_{2(\infty)}$, and $p^+{}_{c(\infty)}$ must all be constants. Equation (85) consequently implies

$$\frac{d\,P^1_{(\infty)}}{d\chi} = \frac{d\,P^2_{(\infty)}}{d\chi} \quad (86)$$

This allows us to eliminate the pressures between Equations (83) and (84) to obtain

$$u^1_{(\infty)} = \frac{s_{(\infty)}\,B_{1(\infty)}\,u^2_{(\infty)}}{N_{(1)}\,[1 - s_{(\infty)}]\,\beta_{2(\infty)}} \quad (87)$$

We may furthermore integrate Equation (84) and use Equation (85) to write

$$P^2_{(\infty)} = \frac{-\,u^2_{(\infty)}\,\chi}{\psi\,[1 - s_{(\infty)}]\,\beta_{2(\infty)}} + C_{(2)} \quad (88)$$

and

$$P^1_{(\infty)} = \frac{-\,u^2_{(\infty)}\,\chi}{\psi\,[1 - s_{(\infty)}]\,\beta_{2(\infty)}} + C_{(2)} + N_{(2)}\,p^+{}_{c(\infty)} \quad (89)$$

Here $C_{(2)}$ is another constant which must be determined.

For phase 2 at $\chi = 0$, we have from Equation (61)

$$u^2_{(\infty)} = 1 + \psi\,c\,[s_{(0)} - s_{(\infty)}] \quad (90)$$

Equations (61), (87), and (90) may be used to write a jump mass balance for phase 1 at $\chi = 0$ as

$$c = \left\{ \psi\,[s_{(\infty)} - s_{(0)}] \left[\frac{[1 - s_{(\infty)}]\,\beta_{2(\infty)}\,N_{(1)}}{s_{(\infty)}\,\beta_{1(\infty)}} + 1 \right] \right\}^{-1} \quad (91)$$

The quantities $u^1_{(\infty)}$, $u^2_{(\infty)}$, and c are completely determined by Equations (87), (90), and (91), inasmuch as $s_{(0)}$ has already been found by means of Equation (80).

Equations (64), (81), and (88) may be used to write a jump momentum balance for phase 2 at $\chi = 0$ as

$$C_{(2)} = \frac{[1 - s_{(0)}]}{[1 - s_{(\infty)}]}\,C_{(1)} \quad (92)$$

Equations (64), (82), (89), and (92) consequently allow us to express the jump momentum balance for phase 1 as

$$C_{(1)} = N_{(2)}\,[s_{(\infty)}\,p^+{}_{c(\infty)} - s_{(0)}\,p^+{}_{c(0)}]\,\frac{[1 - s_{(\infty)}]}{[s_{(0)} - s_{(\infty)}]} \quad (93)$$

We conclude that all required boundary conditions and balance equations are satisfied by a solution of the form assumed in Equations (78) and (79). We have not, however, shown that the solution described here is either unique or stable.

The same problem has not been solved by using Muskat's equation, Equation (25), except in the limit where the capillary pressure function is identically zero (the Buckley-Leverett limit, *13*, *12*, p. 224, see also appendix). There is very little difference between the two solutions for this limiting case; any distinctions are attributable to the different definitions for relative permeability [see Equation (56)]. We have shown in the appendix that a shock wave solution to Equation (25) does not exist for the general case of this problem.

FURTHER WORK

Does an analysis based upon Equation (24) (recommended here) do a better job of describing experimental data than an analysis based upon Muskat's extension of Darcy's law, Equation (25)?

The problems discussed here unfortunately do not provide an answer. They are offered with the hope that they may provide a useful introduction to further work.

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NOTATION

- c = defined by Equation (77)
- $\mathbf{f}$ = external force per unit mass
- $\mathbf{g}_i$ = force per unit volume which phase i exerts upon the pore walls and the other fluid phases contained within S beyond the hydrostatic force and beyond any force attributable to the ambient pressure
- $\hat{\mathbf{g}}_i$ = function defined either by Equation (15) or by Equation (19)
- $\mathbf{I}$ = identity tensor
- L = characteristic length
- $N_{(1)}$, $N_{(2)}$ = dimensionless groups defined by Equations (42)
- p = pressure
- p_0 = reference or ambient pressure
- p_c = capillary pressure; defined by Equation (11)
- $p_c{}^+$ = dimensionless capillary pressure function; see Equations (12) and (38)
- P^i = dimensionless pressure associated with phase i defined by Equation (38)
- $\overline{\mathcal{P}}^{\,i}$ = local volume averaged modified pressure associated with phase i; defined by Equation (9)
- $Q(t)$ = time dependent orthogonal transformation
- R_i = coefficient in Equation (18)
- s_i = local saturation of phase i; see Equation (6)
- S = arbitrary closed surface which is associated with every point in the porous medium
- S_i = closed boundary surface of V_i
- S_{ei} = portion of S_i which coincides with S
- S_{fi} = portion of S_i which coincides with the pore walls
- S_{mi} = portion of S_i which coincides with the boundaries

of the other moving and deforming phases present

- t = time
- $\mathbf{T}$ = stress tensor
- $\overline{\mathbf{u}}$ = local volume averaged velocity of the solid; defined by Equation (16)
- u^i = dimensionless component of velocity associated with phase i; defined by Equation (38)
- $\mathbf{v}$ = velocity
- V = volume of S
- V_i = volume of pores containing phase i which are enclosed by the surface S

Greek Letters

- α_i = Muskat's relative permeability of phase i; replace R_i by L_i in definition of β_i
- β_i = relative permeability of phase i; see Equation (42)
- η = dimensionless coordinate defined by Equation (38)
- θ = contact angle between the fluid-fluid phase interface and the solid composing the pore walls
- Θ = dimensionless time defined by Equation (70)
- μ_i = viscosity of phase i
- ρ = density
- ρ_i = density of phase i
- σ = surface tension
- φ = external force potential; see Equation (7)
- $\varphi_{(j)i}$ = coefficients in Equation (21)
- χ = defined by Equation (76)
- ψ = porosity; see Equation (5)

Special Symbols

- $\overline{}^{\,i}$ = average over V of a quantity associated with phase i; see Equation (3)
- $< >^i$ = an average over V_i of a quantity associated with phase i; see Equation (13)
- (0) = a quantity associated with the region $\chi < 0$
- (∞) = a quantity associated with the region $\chi > 0$

LITERATURE CITED

1. Slattery, J. C., *AIChE J.*, 13, 1066 (1967).
2. *Ibid.*, 15, 866 (1969).
3. *Ibid.*, 14, 50 (1968).
4. Haines, W. B., *J. Agr. Sci.*, 20, 283 (1930).
5. Slattery, J. C. *AIChE J.*, 5, 831 (1965).
6. *Ibid.*, 14, 516 (1968).
7. Truesdell, C., "The Elements of Continuum Mechanics," Springer-Verlag, New York (1966).
8. ———, and W. Noll, "Handbuch der Physik," Vol. 3/3, S. Flügge, ed., Springer-Verlag, Berlin, Germany (1965).
9. Spencer, A. J., and R. S. Rivlin, *Arch. Rational Mechanics and Analysis*, 4, 214 (1959/60).
10. Smith, G. F., *ibid.*, 18, 282 (1965).
11. Muskat, M., R. D. Wyckoff, H. G. Botset, and M. W. Meres, *Trans. Am. Inst. Min. Metall. Engrs.*, 123, 69 (1937).
12. Scheidegger, A. E., "The Physics of Flow through Porous Media," Macmillan, New York (1960).
13. Buckley, S. E., and .M C. Leverett, *Trans. Am. Inst. Min. Metall. Engrs.*, 146, 107 (1942).
14. Truesdell, C., and R. A. Toupin, "Handbuch der Physik," Vol. 3/1, S. Flügge, ed., Springer-Verlag, Berlin, Germany (1960).
15. Slattery, J. C., *Ind. Eng. Chem. Fundamentals*, 6, 108 (1967).
16. Osoba, J. S., J. G. Richardson, J. K. Kerver, J. A. Hafford, and P. M. Blair, *Trans. Am. Inst. Min. Metall. Engrs.*, 192, 47 (1951).

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APPENDIX

We wish to discuss here the displacement problem of the text from the viewpoint of the momentum balance suggested by Muskat, Equation (25). Our intention is to show that a solution in which a surface of discontinuity propagates through a porous medium is inconsistent with Equation (25). Our approach is to show that the assumption of a solution of the form of Equations (78) and (79) leads to a contradiction.

The problem is unchanged from that of the text with the exceptions that Equations (73) and (74) are replaced by

$$\frac{\partial P^1}{\partial \eta} + \frac{N_{(1)}\, u^1}{\alpha_1} = 0 \tag{A1}$$

and

$$\frac{\partial P^2}{\partial \eta} + \frac{u^2}{\alpha_2} = 0 \tag{A2}$$

We use α_i to denote the relative permeability to be used with Muskat's Equation (25). The relation between Muskat's relative permeability α_i and that recommended in this paper is given by Equation (56).

Following the discussion in the text, we may simply replace Equations (80-2), (87-9), and (91), respectively, by

$$\alpha_{1(0)} = 0 \tag{A3}$$

$$P^2{}_{(0)} = \frac{-\chi}{\alpha_{2(0)}} + C_{(1)} \tag{A4}$$

$$P^1{}_{(0)} = \frac{-\chi}{\alpha_{2(0)}} + C_{(1)} + N_{(2)}\ p^+{}_{c(0)} \tag{A5}$$

$$u^1{}_{(\infty)} = \frac{\alpha_{1(\infty)}\ u^2{}_{(\infty)}}{N_{(1)}\ \alpha_{2(\infty)}} \tag{A6}$$

$$P^2{}_{(\infty)} = \frac{-u^2{}_{(\infty)}\ \chi}{\alpha_{2(\infty)}} + C_{(2)} \tag{A7}$$

$$P^1{}_{(\infty)} = \frac{-u^2{}_{(\infty)}\ \chi}{\alpha_{2(\infty)}} + C_{(2)} + N_{(2)}\ p^+{}_{c(\infty)} \tag{A8}$$

and

$$c = \left\{ \psi\,[s_{(\infty)} - s_{(0)}] \left[\frac{\alpha_{2(\infty)}\ N_{(1)}}{\alpha_{1(\infty)}} + 1 \right] \right\}^{-1} \tag{A9}$$

Equation (90) for $u^2{}_{(\infty)}$ applies here without change.

In place of the jump momentum balance, Equation (64), we have here

$$[<\mathcal{P} - p_0>^i] = 0 \tag{A10}$$

When we apply Equation (A10) to phase 2 at $\chi = 0$, we find that

$$\text{at } \chi = 0\text{: } P^2{}_{(0)} = P^2{}_{(\infty)} \tag{A11}$$

or, from Equations (A4) and (A7)

$$C_{(1)} = C_{(2)} \tag{A12}$$

Equation (A10) applied to phase 1 at $\chi = 0$ requires that

$$\text{at } \chi = 0\text{: } P^1{}_{(0)} = P^1{}_{(\infty)} \tag{A13}$$

From Equations (A5), (A8), and (A13), we have

$$p^+{}_{c(0)} = p^+{}_{c(\infty)} \qquad \text{A(14)}$$

which implies that

$$s_{(0)} = s_{(\infty)} \tag{A15}$$

But this contradicts our original assumption of a surface singular with respect to saturation propagating through the permeable structure. We conclude that there is no shock wave solution for this displacement problem which is consistent with Equation (25).

This conclusion is based upon the premise that the capillary pressure function is not identically zero. If we take the capillary function to be identically zero, we have the problem considered by Buckley and Leverett (*13*, *12*, p. 224). For this case, the jump momentum balance, Equation (A10), applied to phase 1 at $\chi = 0$ is identically satisfied without the restriction of Equation (A15).